Fundamentals of Plane Geometry and Intermediate Algebra

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TEXTBOOK REMIXED BY

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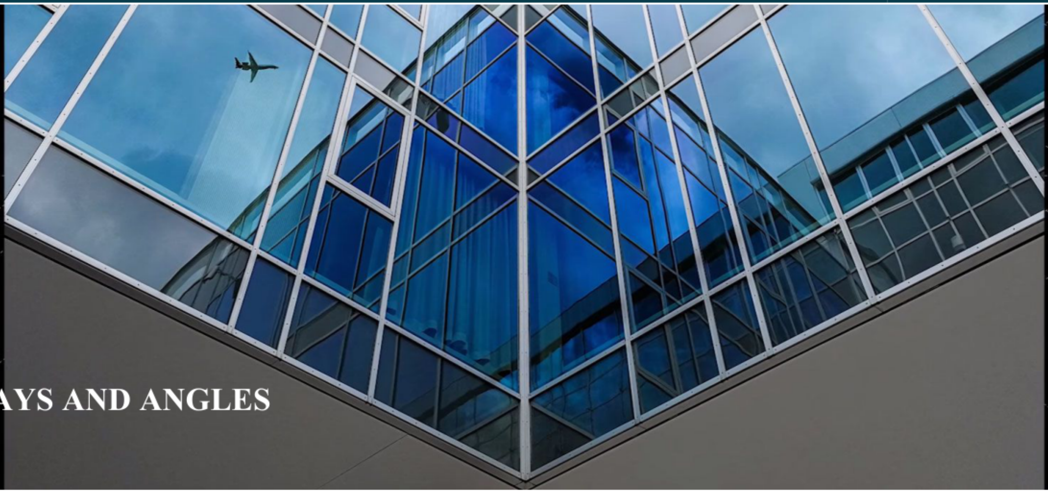
RAYS AND ANGLES



An example of parallel and perpendicular lines as used in symmetric building design. (credit: https://www.diyphotography.net)

Introduction

In this chapter, you will review basic concepts and definitions in geometry such as the different types of angles, and the relationship between the angles that are formed when two parallel lines are cut by a transversal line. You will also review the Pythagorean Theorem and the relationship between the sides of special right triangles.

1.1 Rays and Angles

Introduction

Rays

A ray is a part of a line with exactly one endpoint that extends infinitely in one direction. Rays are named by their endpoint and a point on the ray.

The ray above is called $\overrightarrow{AB}$. The first letter in the ray's name is always the endpoint of the ray, it doesn't matter which direction the ray points.

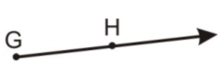
Rays can represent a number of different objects in the real world. For example, the beam of light extending from a flashlight that continues forever in one direction is a ray. The flashlight would be the endpoint of the ray, and the light continues as far as you can imagine so it is the infinitely long part of the ray. Are there other real-life objects that can be represented as rays?

Example 1

Which of the figures below shows $\overrightarrow{GH}$?

A. 

C. 

B. 

D. 

Remember that a ray has one endpoint and extends infinitely in one direction. Choice A is a line segment since it has two endpoints. Choice B has one endpoint and extends infinitely in one direction, so it is a ray. Choice C has no endpoints and extends infinitely in two directions — it is a line. Choice D also shows a ray with endpoint H. Since we need to identify $\overrightarrow{GH}$ with endpoint G, we know that choice B is correct.

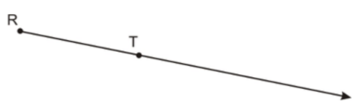
Example 2

Use this space to draw $\overrightarrow{RT}$.

Remember that you are not expected to be an artist. In geometry, you simply need to draw figures that accurately represent the terms in question. This problem asks you to draw a ray. Begin with a line segment. Use your ruler to draw a straight line segment of any length.

Now draw an endpoint on one end and an arrow on the other.

Finally, label the endpoint R and another point on the ray T.

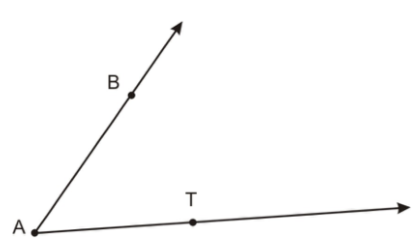


The diagram above shows $\overrightarrow{RT}$.

Angles

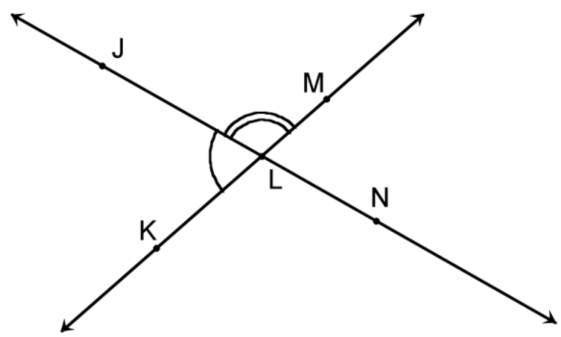
An **angle** is formed when two rays share a common endpoint. That common endpoint is called the **vertex** and the two rays are called the **sides** of the angle. In the diagram below, $\overrightarrow{AB}$ and $\overrightarrow{AT}$ form an angle, $\angle BAT$, or $\angle A$ for short. The symbol $\angle$ is used for naming angles.

The same basic definition for angle also holds when lines, segments, or rays intersect.



Notation Notes:

1. Angles can be named by a number, a single letter at the vertex, or by the three points that form the angle. When an angle is named with three letters, the middle letter will always be the vertex of the angle. In the diagram above, the angle can be written $\angle BAT$, or $\angle TAB$, or $\angle A$. You can use one letter to name this angle since point A is the vertex and there is only one angle at point A.
2. If two or more angles share the same vertex, you MUST use three letters to name the angle. For example, in the image below it is unclear which angle is referred to by $\angle L$. To talk about the angle with one arc, you would write $\angle KLJ$. For the angle with two arcs, you'd write $\angle JLM$.

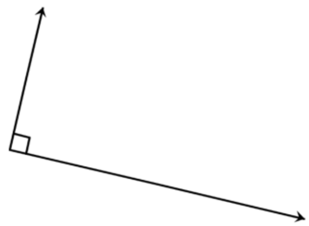


We use a ruler to measure segments by their *length*. But how do we measure an angle? The length of the sides does not change how wide an angle is "open." Instead of using length, the size of an angle is measured by the amount of *rotation* from one side to another. By definition, a full turn is defined as 360 degrees. Use the symbol ° for degrees. You may have heard "360" used as slang for a "full circle" turn, and this expression comes from the fact that a full rotation is 360°.

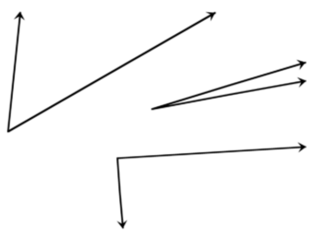
The angle that is made by rotating through one-fourth of a full turn is very special. It measures $\frac{1}{4} \times 360^\circ = 90^\circ$ and we call this a **right angle.** Right angles are easy to identify, as they look like the corners of most buildings, or a corner of a piece of paper.

A **right angle** measures exactly 90°.

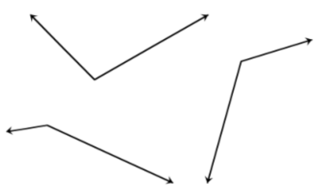
Right angles are usually marked with a small square. When two lines, two segments, or two rays intersect at a right angle, we say that they are **perpendicular**. The symbol $\perp$ is used for two perpendicular lines.



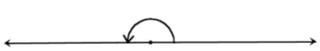
An **acute angle** measures between 0° and 90°.



An **obtuse angle** measures between 90° and 180°.



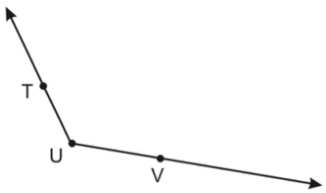
A **straight angle** measures exactly 180°. These are easy to spot since they look like straight lines.



You can use this information to classify any angle you see.

Example 3

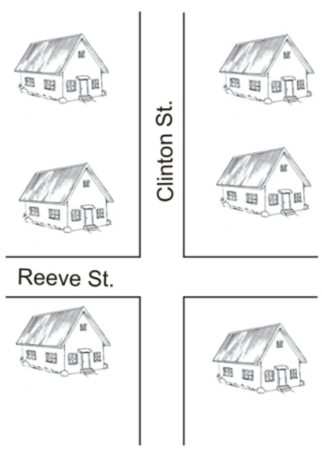
What is the name and classification of the angle below?



Begin by naming this angle. It has three points labeled and the vertex is U. So, the angle will be named $\angle TUV$ or just $\angle U$. For the classification, compare the angle to a right angle. $\angle TUV$ opens wider than a right angle, and less than a straight angle. So, it is **obtuse**.

Example 4

What term best describes the angle formed by Clinton and Reeve streets on the map below?

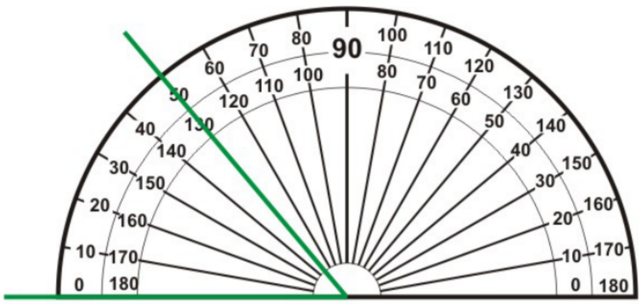


The intersecting streets form a **right angle**. It is a square corner, so it measures 90°.

Notation Note: When we talk about the measure of an angle, we use the symbols $m\angle$. So for example, if we used a protractor to measure $\angle TUV$ and we found that it measured 120°, we could write $m\angle TUV = 120^\circ$.

Example 5

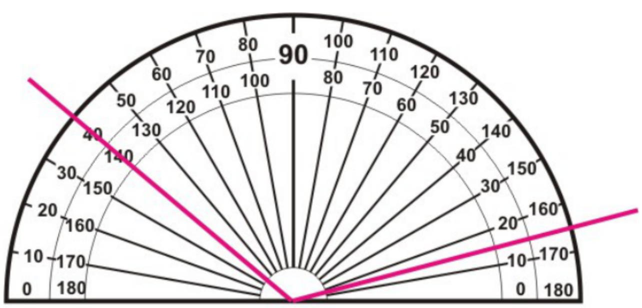
What is the measure of the angle shown below?



This angle is lined up with a protractor at 0°, so you can simply read the final number on the protractor itself. Remember you can check that you are using the correct scale by making sure your answer fits your angle. If the angle is acute, the measure of the angle should be less than 90°. If it is obtuse, the measure will be greater than 90°. In this case, the angle is acute, so its measure is 50°.

Example 6

What is the measure of the angle shown below?



This angle is not lined up with the zero mark on the protractor, so you will have to use subtraction to find its measure.

Using the inner scale, we get $|140 - 15| = |125| = 125^\circ$.

Using the outer scale, $|40 - 165| = |-125| = 125^\circ$.

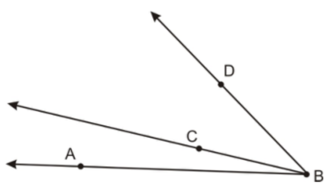
Notice that it does not matter which scale you use. The measure of the angle is 125°.

Multimedia Link The following applet gives you practice measuring angles with a protractor Measuring Angles Applet.

Angle Addition Postulate

You have already encountered the ruler postulate and the protractor postulate. There is also a postulate about angles that is similar to the Segment Addition Postulate.

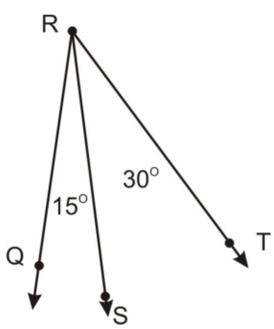
Angle Addition Postulate: The measure of any angle can be found by adding the measures of the smaller angles that comprise it. In the diagram below, if you add $m\angle ABC$ and $m\angle CBD$, you will have found $m\angle ABD$.



Use this postulate just as you did the segment addition postulate to identify the way different angles combine.

Example 7

What is $m\angle QRT$ *in the diagram below?*



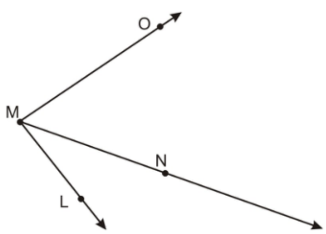
You can see that $m\angle QRS$ is 15°. You can also see that $m\angle SRT$ is 30°. Using the angle addition postulate, you can add these values together to find the total $m\angle QRT$.

$$15 + 30 = 45$$

So, $m\angle QRT$ is 45°.

Example 8

What is $m\angle LMN$ in the diagram below given $m\angle LMO = 85^\circ$ and $m\angle NMO = 53^\circ$?



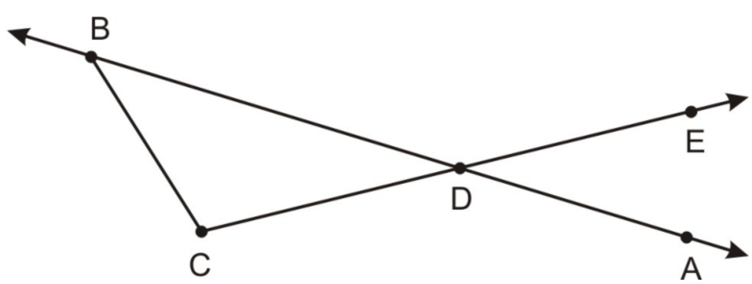
To find $m\angle LMN$, you must subtract $m\angle NMO$ from $m\angle LMO$.

$$85 - 53 = 32$$

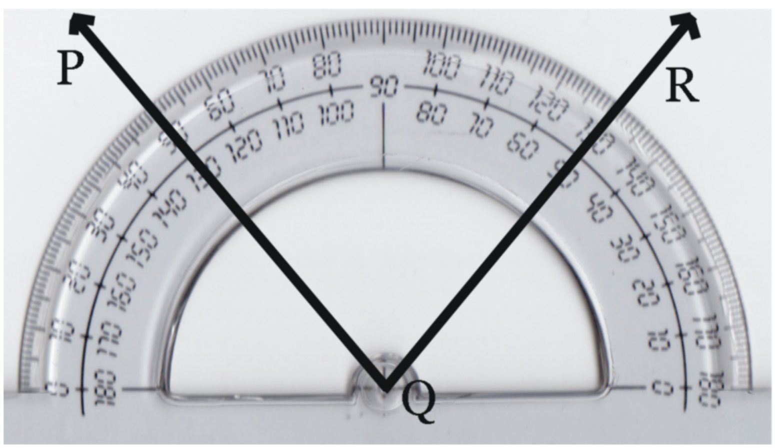
So, $m\angle LMN = 32^\circ$.

Review Questions

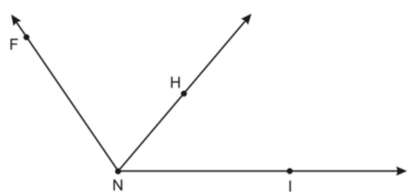
Use this diagram for questions 1-4.



1. Give two possible names for the ray in the diagram.
2. Give four possible names for the line in the diagram.
3. Name an acute angle in the diagram.
4. Name an obtuse angle in the diagram.
5. Name a straight angle in the diagram.
6. Which angle can be named using only one letter?
7. Explain why it is okay to name some angles with only one angle, but with other angles this is not okay.
8. Use a protractor to find $m\angle PQR$ below:



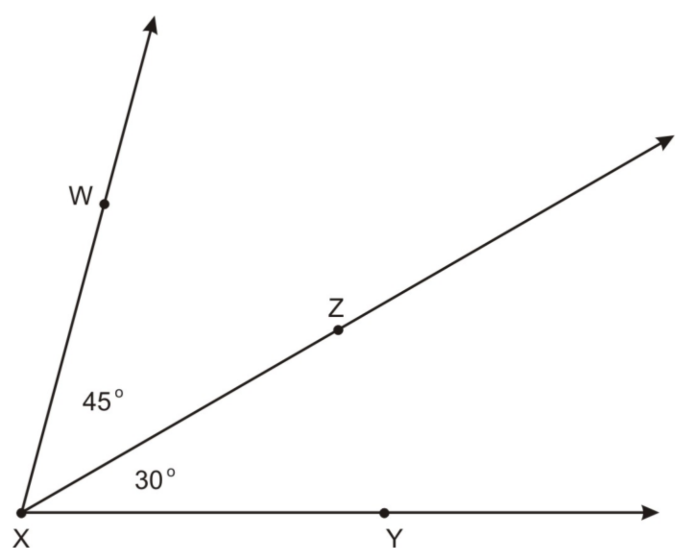
9. Given $m\angle FNI = 125^\circ$ and $m\angle HNI = 50^\circ$, find $m\angle FNH$.



10. True or false: Adding two acute angles will result in an obtuse angle. If false, provide a counterexample.

Review Answers

1. CD or CE
2. BD, DB, AB, or BA are four possible answers. There are more (how many?)
3. BDC
4. BDE or BCD or CDA
5. BDA
6. Angle C
7. If there is more than one angle at a given vertex, then you must use three letters to name the angle. If there is only one angle at a vertex (as in angle C above) then it is permissible to name the angle with one letter.
8. $|(50-130)| = |(-80)| = 80$.
9. $m\angle FNH = |125-50| = |75| = 75^\circ$.
10. False. For a counterexample, suppose two acute angles measure 30° and 45°, then the sum of those angles is 75°, but 75° is still acute. See the diagram for a counterexample:

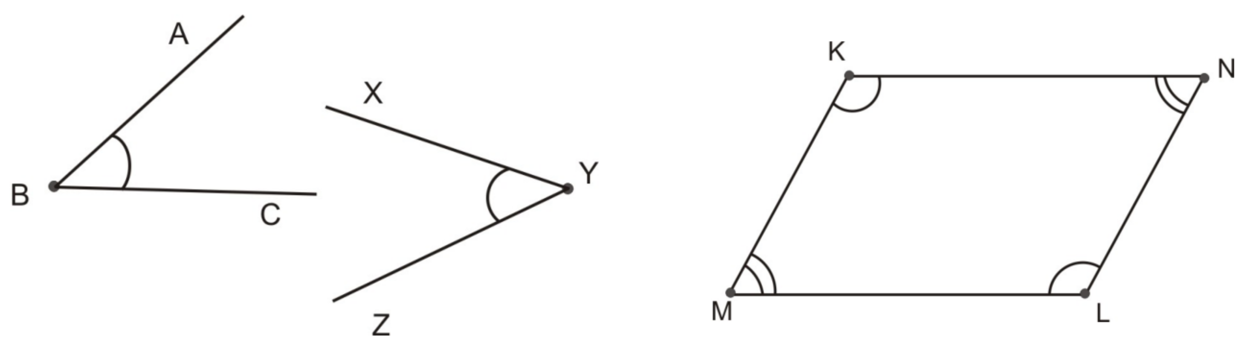


Congruent Angles

You already know that congruent line segments have exactly the same length. You can also apply the concept of congruence to other geometric figures. When angles are congruent, they have exactly the same measure. They may point in different directions, have different side lengths, have different names or other attributes, but their measures will be equal.

Notation Notes:

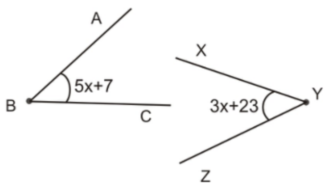
1. When writing that two angles are congruent, we use the congruent symbol: $\angle ABC \cong \angle ZYX$. Alternatively, the symbol $m\angle ABC$ refers to the measure of $\angle ABC$, so we could write $m\angle ABC = m\angle ZYX$ and that has the same meaning as $\angle ABC \cong \angle ZYX$. You may notice then, that *numbers* (such as measurements) are equal while *objects* (such as angles and segments) are congruent.
2. When drawing congruent angles, you use an arc in the middle of the angle to show that two angles are congruent. If two different pairs of angles are congruent, use one set of arcs for one pair, then two for the next pair and so on.



Use algebra to find a way to solve the problem below using this information.

Example 4

The two angles shown below are congruent.



What is the measure of each angle?

This problem combines issues of both algebra and geometry, so make sure you set up the problem correctly. It is given that the two angles are congruent, so they must have the same measurements. Therefore, you can set up an equation in which the expressions representing the angle measures are equal to each other.

$$5x + 7 = 3x + 23$$

Now that you have an equation with one variable, you can solve for the value of x.

$$\begin{aligned} 5x + 7 &= 3x + 23 \\ 5x - 3x &= 23 - 7 \\ 2x &= 16 \\ x &= 8 \end{aligned}$$

So, the value of x is 8. You are not done, however. Use this value of x to find the measure of one of the angles in the problem.

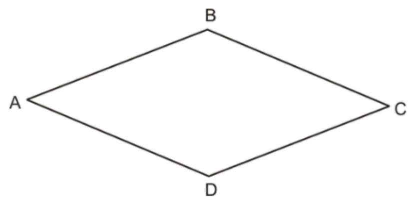
$$\begin{aligned} m\angle ABC &= 5x + 7 \\ &= 5(8) + 7 \\ &= 40 + 7 \\ &= 47 \end{aligned}$$

Finally, we know $m\angle ABC = m\angle XYZ$, so both of the angles measure 47°.

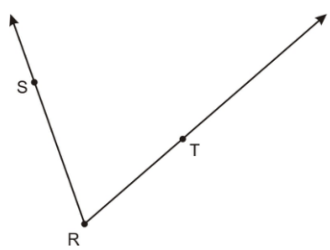
Review Questions

1. Copy the figure below and label it with the following information:

   (a) $\angle A \cong \angle C$
   (b) $\angle B \cong \angle D$
   (c) $\overline{AB} \cong \overline{AD}$

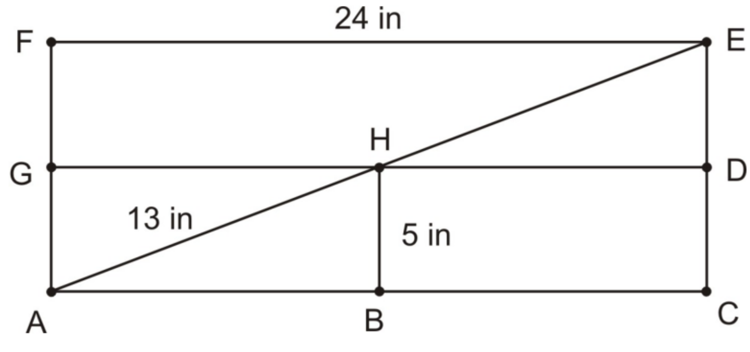


2. Sketch and label an angle bisector $\overrightarrow{RU}$ of $\angle SRT$ below.



3. If we know that $m\angle SRT = 64^\circ$, what is $m\angle SRU$?

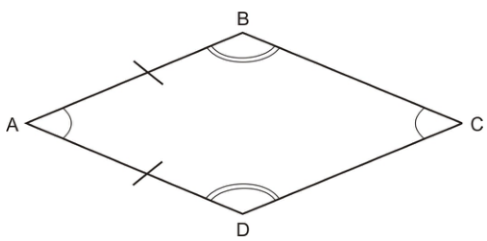
Use the following diagram of rectangle $ACEF$ for questions 4-10. (For these problems you can assume that opposite sides of a rectangle are congruent—later you will prove this is true.)

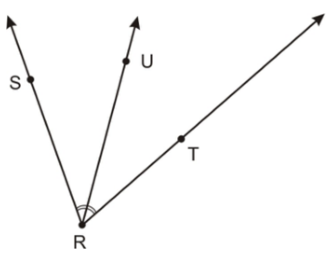


Given that H is the midpoint of $\overline{AE}$ and $\overline{DG}$, find the following lengths:

4. $GH =$
5. $AB =$
6. $AC =$
7. $HE =$
8. $AE =$
9. $CE =$
10. $GF =$
11. How many copies of $\triangle ABH$ can fit inside rectangle $ACEF$?

Review Answers

1. 

2. 

3. 32°
4. $GH = 12$ in
5. $AB = 12$ in
6. $AC = 24$ in
7. $HE = 12$ in
8. $AE = 26$ in
9. $CE = 10$ in
10. $GF = 5$ in
11. 8

1.2 Angle Pairs

Learning Objectives

- Understand and identify complementary angles.
- Understand and identify supplementary angles.
- Understand and utilize the Linear Pair Postulate.
- Understand and identify vertical angles.

Introduction

In this lesson you will learn about special angle pairs and prove the vertical angles theorem, one of the most useful theorems in geometry.

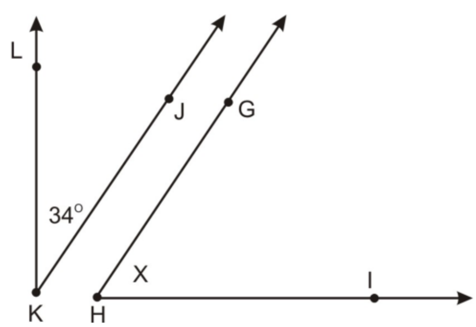
Complementary Angles

A pair of angles are **Complementary angles** if the sum of their measures is 90°.

Complementary angles do not have to be congruent to each other. Rather, their only defining quality is that the sum of their measures is equal to the measure of a right angle: 90°. If the outer rays of two adjacent angles form a right angle, then the angles are complementary.

Example 1

The two angles below are complementary. $m\angle GHI = x$. *What is the value of* x*?*



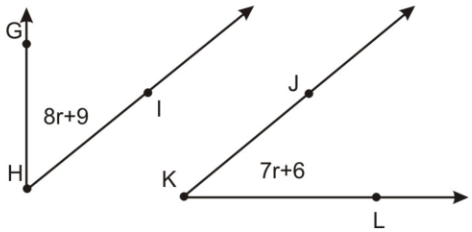
Since you know that the two angles must sum to 90°, you can create an equation. Then solve for the variable. In this case, the variable is x.

$$\begin{aligned} 34 + x &= 90 \\ 34 + x - 4 &= 90 - 34 \\ x &= 56 \end{aligned}$$

Thus, the value of x is 56°.

Example 2

The two angles below are complementary. What is the measure of each angle?



This problem is a bit more complicated than the first example. However, the concepts are the same. If you add the two angles together, the sum will be 90°. So, you can set up an algebraic equation with the values presented.

$$(7r + 6) + (8r + 9) = 90$$

The best way to solve this problem is to solve the equation above for r. Then, you must substitute the value for r back into the original expressions to find the value of each angle.

$$\begin{aligned}(7r + 6) + (8r + 9) &= 90\\ 15r + 15 &= 90\\ 15r + 15 - 15 &= 90 - 15\\ 15r &= 75\\ \frac{15r}{15} &= \frac{75}{15}\\ r &= 5\end{aligned}$$

The value of r is 5. Now substitute this value back into the expressions to find the measures of the two angles in the diagram.

$$\begin{array}{ll} 7r + 6 & 8r + 9\\ 7(5) + 6 & 8(5) + 9\\ 35 + 6 & 40 + 9\\ 41 & 49 \end{array}$$

$m\angle JKL = 41^\circ$ and $m\angle GHI = 49^\circ$. You can check to make sure these numbers are accurate by verifying if they are complementary.

$$41 + 49 = 90$$

Since these two angle measures sum to 90°, they are complementary.

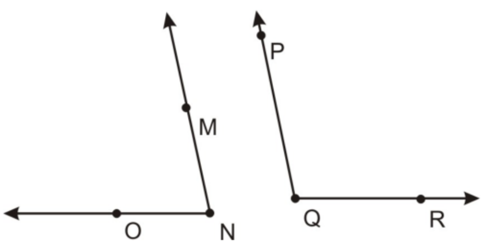
Supplementary Angles

Two angles are **supplementary** if their measures sum to 180°.

Just like complementary angles, supplementary angles need not be congruent, or even touching. Their defining quality is that when their measures are added together, the sum is 180°. You can use this information just as you did with complementary angles to solve different types of problems.

Example 3

The two angles below are supplementary. If $m\angle MNO = 78^\circ$, what is $m\angle PQR$?



This process is very straightforward. Since you know that the two angles must sum to 180°, you can create an equation. Use a variable for the unknown angle measure and then solve for the variable. In this case, let's substitute y for $m\angle PQR$.

$$\begin{aligned} 78 + y &= 180 \\ 78 + y - 78 &= 180 - 78 \\ y &= 102 \end{aligned}$$

So, the measure of $y = 102$ and thus $m\angle PQR = 102^\circ$.

Example 4

What is the measure of two congruent, supplementary angles?

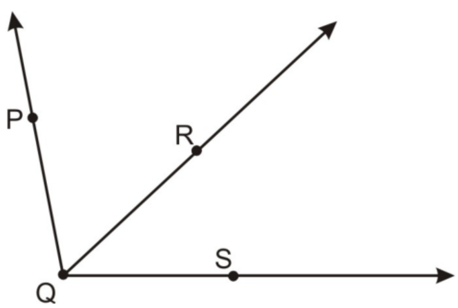
There is no diagram to help you visualize this scenario, so you'll have to imagine the angles (or even better, draw it yourself by translating the words into a picture!). Two supplementary angles must sum to 180°. Congruent angles must have the same measure. So, you need to find two congruent angles that are supplementary. You can divide 180° by two to find the value of each angle.

$$180 \div 2 = 90$$

Each congruent, supplementary angle will measure 90°. In other words, they will be right angles.

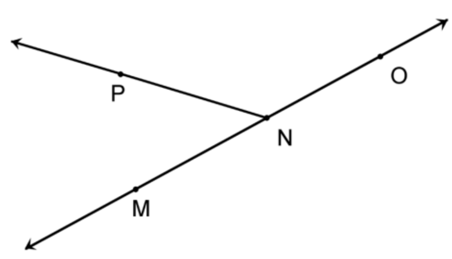
Linear Pairs

Before we talk about a special pair of angles called **linear pairs**, we need to define **adjacent angles.** Two angles are adjacent if they share the same vertex and one side, but they do not overlap. In the diagram below, $\angle PQR$ and $\angle RQS$ are adjacent.



However, $\angle PQR$ and $\angle PQS$ are not adjacent since they overlap (i.e. they share common points in the interior of the angle).

Now we are ready to talk about linear pairs. A **linear pair** is two angles that are adjacent and whose non-common sides form a straight line. In the diagram below, $\angle MNP$ and $\angle PNO$ are a linear pair. Note that $\overleftrightarrow{MO}$ is a line.

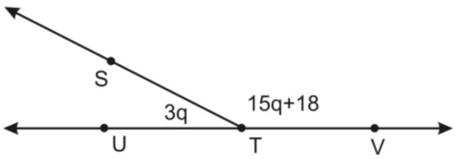


Linear pairs are so important in geometry that they have their own postulate.

Linear Pair Postulate: If two angles are a linear pair, then they are supplementary.

Example 5

The two angles below form a linear pair. What is the value of each angle?



If you add the two angles, the sum will be 180°. So, you can set up an algebraic equation with the values presented.

$$(3q) + (15q + 18) = 180$$

The best way to solve this problem is to solve the equation above for q. Then, you must plug the value for q back into the original expressions to find the value of each angle.

$$\begin{aligned}(3q) + (15q + 18) &= 180\\ 18q + 18 &= 180\\ 18q &= 180 - 18\\ 18q &= 162\\ \frac{18q}{18} &= \frac{162}{18}\\ q &= 9\end{aligned}$$

The value of q is 9. Now substitute this value back into the expressions to determine the measures of the two angles in the diagram.

$$\begin{array}{rr} 3q & 15q + 18 \\ 3(9) & 15(9) + 18 \\ 27 & 135 + 18 \\ & 153 \end{array}$$

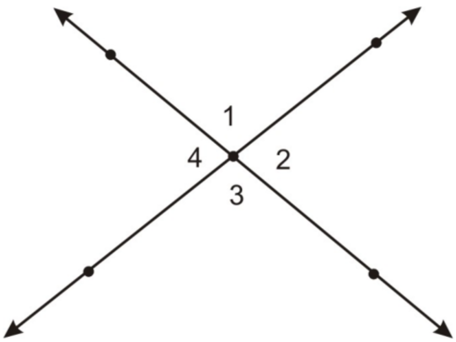
The two angles in the diagram measure 27° and 153°. You can check to make sure these numbers are accurate by verifying if they are supplementary.

$$27 + 153 = 180$$

Vertical Angles

Now that you understand supplementary and complementary angles, you can examine more complicated situations. Special angle relationships are formed when two lines intersect, and you can use your knowledge of linear pairs of angles to explore each angle further.

Vertical angles are defined as two non-adjacent angles formed by intersecting lines. In the diagram below, $\angle 1$ and $\angle 3$ are vertical angles. Also, $\angle 4$ and $\angle 2$ are vertical angles.



Suppose that you know $m\angle 1 = 100^\circ$. You can use that information to find the measurement of all the other angles. For example, $\angle 1$ and$\angle 2$ must be supplementary since they are a linear pair. So, to find $m\angle 2$, subtract 100° from 180°.

$$\begin{aligned} m\angle 1 + m\angle 2 &= 180 \\ 100 + m\angle 2 &= 180 \\ m\angle 2 &= 180 - 100 \\ m\angle 2 &= 80 \end{aligned}$$

So $\angle 2$ measures 80°. Knowing that angles 2 and 3 are also supplementary means that $m\angle 3 = 100^\circ$, since the sum of 100° and 80° is 180°. If angle 3 measures 100°, then the measure of angle 4 must be 80°, since 3 and 4 are also supplementary. Notice that angles 1 and 3 are congruent (100°) and 2 and 4 are congruent (80°).

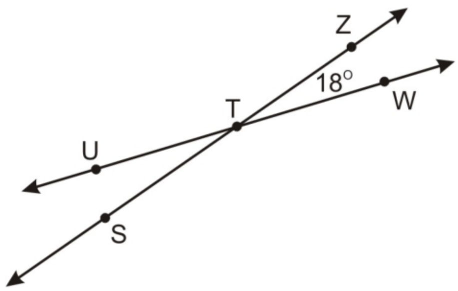
The **Vertical Angles Theorem** states that if two angles are vertical angles then they are congruent.

We can prove the vertical angles theorem using a process just like the one we used above. There was nothing special about the given measure of $\angle 1$. Here is proof that vertical angles will always be congruent: Since $\angle 1$ and $\angle 2$ form a linear pair, we know that they are supplementary: $m\angle 1 + m\angle 2 = 180^\circ$. For the same reason, $\angle 2$ and$\angle 3$ are supplementary: $m\angle 2 + m\angle 3 = 180^\circ$. Using a substitution, we can write $m\angle 1 + m\angle 2 = m\angle 2 + m\angle 3$. Finally, subtracting $m\angle 2$ on both sides yields $m\angle 1 = m\angle 3$. Or, by the definition of congruent angles, $\angle 1 \cong \angle 3$.

Use your knowledge of vertical angles to solve the following problem.

Example 6

What is $m\angle STU$ in the diagram below?

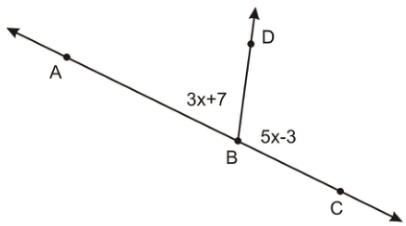


Using your knowledge of intersecting lines, you can identify that $\angle STU$ is vertical to the angle marked 18°. Since vertical angles are congruent, they will have the same measure. So, $m\angle STU$ is also equal to 18°.

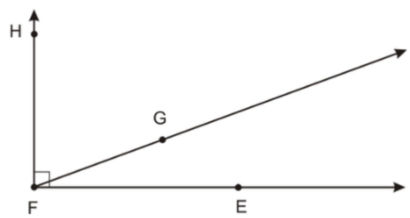
Review Questions

1. Find the measure of the angle complementary to $\angle A$ if $m\angle A =$
   (a) 45°
   (b) 82°
   (c) 19°
   (d) z°
2. Find the measure of the angle supplementary to $\angle B$ if
   (a) 45°
   (b) 118°
   (c) 32°
   (d) x°

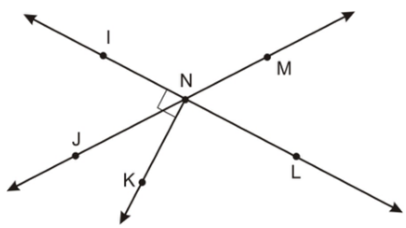
3. Find $m\angle ABD$ and $m\angle DBC$.



4. Given $m\angle EFG = 20^\circ$, Find $m\angle HFG$.



Use the diagram below for exercises 5 and 6. Note that $\overline{NK} \perp \overleftrightarrow{IL}$.



5. Identify each of the following (there may be more than one correct answer for some of these questions).
   (a) Name one pair of vertical angles.
   (b) Name one linear pair of angles.
   (c) Name two complementary angles.
   (d) Nam two supplementary angles.
6. Given that $m\angle IJN = 63^\circ$, find
   (a) $m\angle JNK$.
   (b) $m\angle KNL$.
   (c) $m\angle MNL$.
   (d) $m\angle MNI$.

Review Answers

1. (a) 45°
   (b) 8°
   (c) 81°
   (d) $(90 - z)^\circ$
2. (a) 135°
   (b) 62°
   (c) 148°
   (d) $(180 - x)^\circ$
3. $m\angle ABD = 73^\circ$, $m\angle DBC = 107^\circ$
4. $m\angle HFG = 70^\circ$
5. (a) $\angle JNI$ and $\angle MNL$ (or $\angle INM$ and $\angle JNL$ also works);
   (b) $\angle INM$ and $\angle MNL$ (or $\angle INK$ and $\angle KNL$ also works);
   (c) $\angle INK$ and $\angle JNK$;
   (d) same as (b) $\angle INM$ and $\angle MNL$ (or $\angle INK$ and $\angle KNL$ also works).
6. (a) 27°
   (b) 90°
   (c) 63°
   (d) 117°